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Axiomatizing the monodic fragment of first-order temporal logic

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Abstract

It is known that even seemingly small fragments of the first-order temporal logic over the natural numbers are not recursively enumerable. In this paper we show that the monodic (not monadic, where this result does not hold) fragment is an exception by constructing its finite Hilbert-style axiomatization. We also show that the monodic fragment with equality is not recursively axiomatizable. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The first-order temporal logic over the natural numbers $TL(\mathbb{N})$ and even its two-variable monadic fragment are known to be not recursively enumerable (see e.g. [4,8] and references therein). However, it is shown in [8] that various non-trivial fragments of $TL(\mathbb{N})$ and other first-order temporal logics are decidable. All these fragments operate only with the so-called monodic formulas.

Definition 1 (*Monodic formulas*). Let $\mathcal{TL}$ be the first-order temporal language with the temporal operators $\mathcal{S}$ ('since'), $\mathcal{U}$ ('until'), $\bigcirc$ ('at the next moment'), and $\bigcirc_P$ ('at the previous moment'), but without equality and functional symbols. Denote by $\mathcal{TL}_1$ the set of all $\mathcal{TL}$-formulas φ such that any subformula of φ of the form $\psi_1 \mathcal{S} \psi_2$,

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$\psi_1 \mathcal{U} \psi_2$, $\bigcirc\psi$ or $\bigcirc_P\psi$ has at most one free variable. Such formulas are called *monodic*, and $\mathcal{TL}_1$ is called the *monodic fragment* of $\mathcal{TL}$.

Here are some examples of monodic formulas, where $\Diamond_F$ and $\Diamond_P$ are the operators 'some time in the future' and 'some time in the past' expressible via $\mathcal{U}$ and $\mathcal{S}$ ($\Box_F$ and $\Box_P$ are their duals):

- $\exists x \Diamond_F \varphi(x) \leftrightarrow \Diamond_F \exists x \varphi(x)$ (the Barcan formula);
- $\Box_F \exists x\ (\bigcirc Old(x) \wedge \neg(\top \mathcal{S}\ Old(x)))$ ('at every moment, someone starts to get old');
- $\forall x \Box_F(Sub(x) \rightarrow \bigcirc\Box_F \neg Sub(x))$ (this is a constraint for temporal databases from [2]: 'an order can be submitted only once');
- $\Diamond_P \exists y\ \ Works(x, y) \wedge \neg\exists y\ \ Works(x, y) \wedge \Diamond_F \exists y\ \ Works(x, y)$ (this is a query to a temporal database from [3]: 'list all persons who have been unemployed between jobs').

The following formula (one more query from [3]) is not monodic:

- $\Box_P \Box_F(\neg\exists y(Works(x, y) \wedge \bigcirc\ Works(x, y) \wedge \bigcirc\bigcirc\ Works(x, y)))$ ('find all job-hoppers — people who never spent more than two years in one place').

It turns out that the monodic fragment of $TL(\mathbb{N})$, though undecidable because it contains full first-order logic, is recursively enumerable, and moreover can be axiomatized in a rather natural way. To present such an axiomatization is the main aim of this paper (Section 2). We show, however, that by adding equality to $\mathcal{TL}_1$ we restore the 'status-quo': the monodic fragment of $TL(\mathbb{N})$ with equality becomes not recursively enumerable (Section 3). And, finally, we note that the monodic fluted fragment (in the sense of [9,10]) as well as the monodic loosely guarded fragment [12] of many first-order temporal logics are decidable (Section 4), thus extending the list of decidable fragments from [8].

2. Axiomatization

To make the proofs more transparent, we confine ourselves to considering only the 'future fragment' of the language $\mathcal{TL}$ containing two primitive temporal operators $\bigcirc$ and $\mathcal{U}$. The reader should not have any problems with extending the results to the full language simply by adding the corresponding 'past counterparts'. Besides, for purely technical reasons we slightly change the semantics of $\mathcal{U}$ as compared with [8]:

- $n \models \varphi \mathcal{U} \psi$ iff there exists $k \geqslant n$ such that $k \models \psi$ and $m \models \varphi$ for all m such that $n \leqslant m < k$ (i.e., $m \in [n, k)$).

The 'until' from [8] can be defined as $\bigcirc(\varphi \mathcal{U} \psi)$. (Note that now $\bigcirc$ is not expressible via $\mathcal{U}$.)

$TL(\mathbb{N})$ is the set of $\mathcal{TL}$-formulas that are valid in all first-order temporal models with the flow of time isomorphic to $\langle \mathbb{N}, < \rangle$ and having constant domains and rigid designators (for a detailed definition consult [8]).

We propose the following axiomatization $\mathcal{MON}$ of the monodic fragment of $TL(\mathbb{N})$ (cf. [5,11,4]):

Axiom schemata (over formulas in $\mathcal{TL}_1$)
(cl) the set of axiom schemata from some axiomatization of classical first-order logic;
(n_1) $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$;
(n_2) $\bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi$;
(n_3) $\bigcirc\forall x\varphi \leftrightarrow \forall x\bigcirc\varphi$;
(u_1) $\varphi\mathcal{U}\psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi\mathcal{U}\psi))$.

Inference rules (over formulas in $\mathcal{TL}_1$)
(cl) the rules of the axiomatization of classical first-order logic;
(n_4) $\frac{\varphi}{\bigcirc\varphi}$;
(u_2) $\frac{\chi \to \neg\psi \wedge \bigcirc\chi}{\chi \to \neg(\varphi\mathcal{U}\psi)}$.
(Here and below we assume $\neg$, $\bigcirc$ and $\forall$ to connect stronger than $\wedge$, $\vee$, and $\mathcal{U}$, which in turn are stronger than $\to$ and $\leftrightarrow$.)

Denote by $\vdash$ the consequence relation determined by $\mathcal{MON}$.

In the remaining part of this section we will be proving the following:

Theorem 2. *For every monodic $\mathcal{TL}$-formula φ, we have $\vdash \varphi$ iff $\varphi \in TL(\mathbb{N})$.*

It is easy to check the *soundness* part ($\Rightarrow$) of the theorem. For the only non-standard thing in $\mathcal{MON}$ is the rule (u_2). Suppose $\alpha = \chi \to \neg\psi \wedge \bigcirc\chi$ is in $TL(\mathbb{N})$, but $\beta = \chi \to \neg(\varphi\mathcal{U}\psi)$ is not. Consider a model of $TL(\mathbb{N})$ refuting β. Then there is a moment n such that $n \models \chi$ and $n \models \varphi\mathcal{U}\psi$, i.e., there exists $k \geqslant n$ for which $k \models \psi$ and $m \models \varphi$ for all $m \in [n,k)$. As α is valid in the model, $l \models \chi$ whenever $l \geqslant n$. But then $k \models \neg\psi$, which is a contradiction.

To prove the *completeness* part, we can show that if $\nvdash \varphi$ then there is a model based on $\mathbb{N}$ and refuting φ, or, to put it another way, we can show that if $\nvdash \neg\varphi$ — i.e., φ is *consistent* with $\mathcal{MON}$ — then φ is satisfiable in a model of $TL(\mathbb{N})$. Thus, we need some means of constructing models. As in [8], we will be using for this purpose quasi-models, appropriately modified for the needs of this proof. (To make the paper self-contained we shall repeat the required definitions from [8].)

For a $\mathcal{TL}$-formula φ, let

$$\bigcirc\neg\{\varphi\} = sub\,\varphi \cup \{\neg\psi : \psi \in sub\,\varphi\} \cup \{\bigcirc\neg\psi : \psi \in sub\,\varphi\},$$

where $sub\,\varphi$ is the set of all subformulas of φ. Denote by $sub_n\,\varphi$ the subset of $\bigcirc\neg\{\varphi\}$ containing formulas with $\leqslant n$ free variables. Without loss of generality we may assume that $\bigcirc\neg\{\varphi\}$ is closed under negation, at least modulo the equivalences $\neg\neg\alpha \leftrightarrow \alpha$ and (n_2). By $con\,\varphi$ we denote the set of individual constants occurring in φ. In what follows we will not be distinguishing between a finite set Γ of formulas and the conjunction $\bigwedge\Gamma$ of formulas in it.

Let x be a variable not occurring in φ. Put

$$sub_x\,\varphi = \{\psi\{x/y\} : \psi(y) \in sub_1\,\varphi\}.$$

Definition 3 (*Type*). A *type* for φ is any Boolean-saturated subset t of $sub_x\,\varphi$, i.e.,
- $\psi \wedge \chi \in t$ iff $\psi \in t$ and $\chi \in t$, for every $\psi \wedge \chi \in sub_x\,\varphi$;
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_x\,\varphi$.

We say that two types t and t' *agree on* $sub_0\,\varphi$ if $t \cap sub_0\,\varphi = t' \cap sub_0\,\varphi$. Given a type t for φ and a constant $c \in con\,\varphi$, the pair $\langle t, c\rangle$ is called an *indexed type* for φ (indexed by c).

Definition 4 (*State candidate*). Suppose that T is a set of types for φ that agree on $sub_0\,\varphi$, and T^{con} a set containing, for each $c \in con\,\varphi$, one indexed type $\langle t, c\rangle$ such that $t \in T$. Then the pair $\mathfrak{C} = \langle T, T^{con}\rangle$ is called a *state candidate* for φ. A *pointed state candidate* for φ is a pair $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$, where $\mathfrak{C} = \langle T, T^{con}\rangle$ is a state candidate for φ and t a type in T. We also say that $\mathfrak{P}$ is the *state candidate based on* $\mathfrak{C}$ *with point* t.

Given a state candidate $\mathfrak{C} = \langle T, T^{con}\rangle$ for φ and a pointed state candidate $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$, we put

$$\alpha_{\mathfrak{C}} = \bigwedge_{t \in T} \exists x t(x) \quad \wedge \quad \forall x \bigvee_{t \in T} t(x) \quad \wedge \bigwedge_{\langle t, c\rangle \in T^{con}} t(c),$$

$$\beta_{\mathfrak{P}} = \alpha_{\mathfrak{C}} \wedge t.$$

Say that $\mathfrak{C}$ (or $\mathfrak{P}$) is *consistent* if the formula $\alpha_{\mathfrak{C}}$ (respectively, $\beta_{\mathfrak{P}}$) is consistent with $\mathcal{MON}$.

Definition 5 (*Suitable pairs*). (1) A pair (t_1, t_2) of types for φ is called *suitable* if the formula $t_1 \wedge \bigcirc t_2$ is consistent with $\mathcal{MON}$.

(2) A pair of state candidates $(\mathfrak{C}_1, \mathfrak{C}_2)$ is *suitable* if $\alpha_{\mathfrak{C}_1} \wedge \bigcirc \alpha_{\mathfrak{C}_2}$ is consistent with $\mathcal{MON}$. In this case we write $\mathfrak{C}_1 \prec \mathfrak{C}_2$.

(3) A pair $\mathfrak{P}_1 = \langle \mathfrak{C}_1, t_1\rangle$, $\mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$ of pointed state candidates for φ is called *suitable* if the formula $\beta_{\mathfrak{P}_1} \wedge \bigcirc \beta_{\mathfrak{P}_2}$ is consistent with $\mathcal{MON}$. In this case we write $\mathfrak{P}_1 \prec \mathfrak{P}_2$.

(4) Let $c \in con\,\varphi$. A pair $\mathfrak{P}_1 = \langle \mathfrak{C}_1, t_1\rangle$, $\mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$ of pointed state candidates for φ is called *suitable* for c if it is suitable and $\langle t_1, c\rangle \in T_1^{con}$, $\langle t_2, c\rangle \in T_2^{con}$, where $\mathfrak{C}_i = \langle T_i, T_i^{con}\rangle$, $i = 1, 2$. In this case we write $\mathfrak{P}_1 \prec_c \mathfrak{P}_2$.

Definition 6 (*Run*). Let $\mathfrak{Q} = (\mathfrak{C}_n = \langle T_n, T_n^{con}\rangle : n \in \mathbb{N})$ be a sequence of state candidates for φ. A *run* in $\mathfrak{Q}$ is a map r associating with every $n \in \mathbb{N}$ a type $r(n)$ in T_n in such a way that the following holds:
- the pairs $(r(n), r(n+1))$ are suitable for all $n \in \mathbb{N}$;
- for every $\chi \mathcal{U} \psi \in sub\,\varphi$, we have $\chi \mathcal{U} \psi \in r(n)$ iff there exists $m \geqslant n$ such that $\psi \in r(m)$ and $\chi \in r(k)$ for all $k \in [n, m)$.

Definition 7 (*Quasi-model*). A sequence $\mathfrak{Q} = (\mathfrak{C}_n = \langle T_n, T_n^{con}\rangle : n \in \mathbb{N})$ of state candidates for φ is called a *quasi-model* for φ if
- the pairs $(\mathfrak{C}_n, \mathfrak{C}_{n+1})$ are suitable for all $n \in \mathbb{N}$;
- for every $n \in \mathbb{N}$ and every type t in T_n there exists a run r in $\mathfrak{Q}$ such that $r(n) = t$;

- for every constant c, the function r_c defined by $r_c(n)=t$, for $\langle t,c\rangle \in T_n^{con}$, $n \in \mathbb{N}$, is a run in $\mathfrak{Q}$.

Say that φ is *satisfied* in $\mathfrak{Q}$ if there are $n \in \mathbb{N}$ and a type t in T_n such that $\varphi \in t$.

Lemma 8. *If a monodic sentence φ is satisfied in a quasi-model for φ, then it is satisfiable.*

Proof. The proof is almost the same as the corresponding part of the proof of Theorem 14 in [8]. The only difference is that now we are not given that the state candidates in $\mathfrak{Q}$ are realizable. We know; however, that every state candidate $\mathfrak{C}_i$ is consistent with $\mathcal{MON}$, and so there is a first-order model realizing $\mathfrak{C}_i$ (subformulas of the form $\bigcirc\psi$ and $\chi\mathcal{U}\psi$ that are not in the scope of another temporal operator are treated as unary predicates or propositional variables). The remaining part is precisely the same as that of the proof mentioned above. □

Thus, to prove Theorem 2, it suffices to show that a sentence φ is satisfied in a quasi-model whenever φ is consistent.

The next two lemmas show some properties of suitable pairs.

Lemma 9. (i) *Suppose (t_1,t_2) is a suitable pair of types for φ. If $\bigcirc\psi \in t_1$, then $\psi \in t_2$. If $\chi\mathcal{U}\psi \in t_1$, then either $\psi \in t_1$ or $\chi \in t_1$ and $\chi\mathcal{U}\psi \in t_2$.*

(ii) *Suppose $(\mathfrak{C}_1,\mathfrak{C}_2)$ is a suitable pair of state candidates, $\mathfrak{C}_1=\langle T_1,T_1^{con}\rangle$ and $\mathfrak{C}_2=\langle T_2,T_2^{con}\rangle$. Then*

- *for every $t_1 \in T_1$ there exists a $t_2 \in T_2$ such that the pair (t_1,t_2) is suitable;*
- *for every $t_2 \in T_2$ there exists a $t_1 \in T_1$ such that (t_1,t_2) is suitable, and*
- *if $\langle t_1,c\rangle \in T_1^{con}$ and $\langle t_2,c\rangle \in T_2^{con}$, then the pair (t_1,t_2) is suitable.*

Proof. (i) Suppose $\bigcirc\psi \in t_1$, but $\psi \notin t_2$. Then $\neg\psi \in t_2$. Since $t_1 \wedge \bigcirc t_2$ is consistent (and $\bigcirc$ distributes over $\wedge$), the formula $\bigcirc\psi \wedge \bigcirc\neg\psi$ is also consistent, which is impossible.

Suppose now that $\chi\mathcal{U}\psi \in t_1$. In view of (u_1) and consistency of t_1, we then have either $\psi \in t_1$ or $\chi, \bigcirc(\chi\mathcal{U}\psi) \in t_1$. And as we have just shown, if $\bigcirc(\chi\mathcal{U}\psi) \in t_1$ then $\chi\mathcal{U}\psi \in t_2$.

(ii) Assume that there is $t_1 \in T_1$ such that none of the pairs (t_1,t_2), for $t_2 \in T_2$, is suitable. It follows that

$$\vdash t_1 \rightarrow \bigcirc \bigwedge_{t_2 \in T_2} \neg t_2$$

from which

$$\vdash \exists x t_1 \rightarrow \exists x \bigcirc \bigwedge_{t_2 \in T_2} \neg t_2$$

and so, by (n_2) and (n_3)

$$\vdash \neg\left(\exists x t_1 \wedge \bigcirc\forall x \bigvee_{t_2\in T_2} t_2\right)$$

contrary to $(\mathfrak{C}_1, \mathfrak{C}_2)$ being suitable.

Now suppose that there is $t_2 \in T_2$ such that none of the pairs (t_1, t_2), for $t_1 \in T_1$, is suitable. Then

$$\vdash \exists x \bigcirc t_2 \rightarrow \exists x \neg \bigvee_{t_1\in T_1} t_1,$$

which is equivalent to

$$\vdash \neg\left(\forall x \bigvee_{t_1\in T_1} t_1 \wedge \exists x \bigcirc t_2\right),$$

contrary to $(\mathfrak{C}_1, \mathfrak{C}_2)$ being suitable.

Finally, assume that $\langle t_1, c\rangle \in T_1^{con}$ and $\langle t_2, c\rangle \in T_2^{con}$. Then $t_1(c) \wedge \bigcirc t_2(c)$ is consistent, and so the pair (t_1, t_2) is suitable. $\square$

Lemma 10. (i) *For every consistent state candidate* $\mathfrak{C}_1$ *for* φ, *there is a state candidate* $\mathfrak{C}_2$ *for* φ *such that* $\mathfrak{C}_1 \prec \mathfrak{C}_2$.

(ii) *For every consistent pointed state candidate* $\mathfrak{P}_1 = \langle \mathfrak{C}_1, t_1\rangle$ *and every state candidate* $\mathfrak{C}_2$ *for* φ *such that* $\mathfrak{C}_1 \prec \mathfrak{C}_2$, *there is a pointed state candidate* $\mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$ *for* φ *such that* $\mathfrak{P}_1 \prec \mathfrak{P}_2$.

(iii) *Let* $c \in con\,\varphi$. *For every consistent pointed state candidate* $\mathfrak{P}_1 = \langle \mathfrak{C}_1, t_1\rangle$ *for* φ *with* $\langle t_1, c\rangle \in T_1^{con}$ *and every state candidate* $\mathfrak{C}_2$ *such that* $\mathfrak{C}_1 \prec \mathfrak{C}_2$, *we have* $\mathfrak{P}_1 \prec_c \mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$, *where* $\langle t_2, c\rangle \in T_2^{con}$.

Proof. (i) Denote by ζ_φ the disjunction of formulas $\alpha_{\mathfrak{C}}$, for all state candidates $\mathfrak{C}$ for φ. As ζ_φ is clearly true in all classical first-order models (subformulas of the form $\bigcirc\psi(x)$ or $\chi(x)\mathcal{U}\psi(x)$ are treated as unary predicates), we have $\vdash \zeta_\varphi$ and $\vdash \bigcirc\zeta_\varphi$. Therefore, $\alpha_{\mathfrak{C}_1} \wedge \bigcirc\zeta_\varphi$ is consistent, and so there must be a state candidate $\mathfrak{C}_2$ such that $\alpha_{\mathfrak{C}_1} \wedge \bigcirc\alpha_{\mathfrak{C}_2}$ is consistent.

(ii) is easy; we leave it to the reader as an exercise.

(iii) Suppose that $\langle t_1, c\rangle \in T_1^{con}$ and $\mathfrak{C}_1 \prec \mathfrak{C}_2$. Let $\langle t_2, c\rangle \in T_2^{con}$. Then $\mathfrak{P}_1 \prec_c \mathfrak{P}_2$, where $\mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$, for otherwise we would have

$$\vdash \alpha_{\mathfrak{C}_1} \wedge t_1 \rightarrow \neg\bigcirc(\alpha_{\mathfrak{C}_2} \wedge t_2)$$

and so

$$\vdash \alpha_{\mathfrak{C}_1} \wedge t_1(c) \rightarrow \neg\bigcirc(\alpha_{\mathfrak{C}_2} \wedge t_2(c)),$$

i.e., $\vdash \alpha_{\mathfrak{C}_1} \rightarrow \neg\bigcirc\alpha_{\mathfrak{C}_2}$, which is a contradiction. $\square$

Suppose $\mathfrak{P}_0 = \langle \mathfrak{C}_0, t_0 \rangle$ is a consistent pointed state candidate for φ and $\chi \mathcal{U} \psi \in t_0$. Suppose also that $\mathfrak{P}_0, \dots, \mathfrak{P}_n$, for some $n \geqslant 0$, is a sequence of pointed state candidates $\mathfrak{P}_i = \langle \mathfrak{C}_i, t_i \rangle$ such that

$$\mathfrak{P}_0 \prec \mathfrak{P}_1 \prec \cdots \prec \mathfrak{P}_n$$

and there exists $k \leqslant n$ such that $\psi \in t_k$ and $\chi \in t_i$ for all $i \in [0, k)$. Then we say that this sequence *realizes* $\chi \mathcal{U} \psi$ in t_0. If for some $c \in con\,\varphi$

$$\mathfrak{P}_0 \prec_c \mathfrak{P}_1 \prec_c \cdots \prec_c \mathfrak{P}_n$$

then we say that the sequence $\mathfrak{P}_0, \dots, \mathfrak{P}_n$ *c-realizes* $\chi \mathcal{U} \psi$ in t_0.

Lemma 11. *For every consistent pointed state candidate $\mathfrak{P}_0 = \langle \mathfrak{C}_0, t_0 \rangle$ and every formula $\chi \mathcal{U} \psi \in t_0$, there is a sequence $\mathfrak{P}_0, \dots, \mathfrak{P}_n$ realizing $\chi \mathcal{U} \psi$ in t_0. Moreover, if $\langle t_0, c \rangle \in T_0^{con}$ then we can find a sequence $\mathfrak{P}_0, \dots, \mathfrak{P}_n$ which c-realizes $\chi \mathcal{U} \psi$ in t_0.*

Proof. Suppose otherwise. As $\mathfrak{P}_0$ is consistent, we have

$$\nvdash \beta_{\mathfrak{P}_0} \to \neg(\chi \mathcal{U} \psi). \tag{1}$$

Let $\mathcal{S}$ be the minimal set of pointed state candidates for φ such that

- $\mathfrak{P}_0 \in \mathcal{S}$ and
- if $\mathfrak{D}_1 \in \mathcal{S}$ and $\mathfrak{D}_1 \prec \mathfrak{D}_2$, then $\mathfrak{D}_2 \in \mathcal{S}$.

Consider the (non-empty) disjunction

$$\vartheta = \bigvee_{\mathfrak{D} \in \mathcal{S}} \beta_{\mathfrak{D}}.$$

Note first that

$$\vdash \vartheta \to \neg\psi. \tag{2}$$

Indeed, otherwise the formula $\vartheta \wedge \psi$ is consistent, and so

$$\bigvee_{\mathfrak{D} \in \mathcal{S}} (\beta_{\mathfrak{D}} \wedge \psi)$$

is consistent as well. Hence there is $\mathfrak{D} \in \mathcal{T}$ such that $\beta_{\mathfrak{D}} \wedge \psi$ is consistent, which means, in particular, that ψ is in the point t of $\mathfrak{D}$ (for otherwise $\neg\psi \in t$ and $\beta_{\mathfrak{D}} \wedge \psi$ cannot be consistent). Thus we have a sequence

$$\mathfrak{P}_0 \prec \mathfrak{P}_1 \prec \cdots \prec \mathfrak{P}_n$$

such that $\mathfrak{P}_i = \langle \mathfrak{C}_i, t_i \rangle$ and $\psi \in t_n$. As all pairs (t_i, t_{i+1}), $0 \leqslant i < n$, are suitable, it follows from Lemma 9(i) that the sequence $\mathfrak{P}_0, \dots, \mathfrak{P}_n$ realizes $\chi \mathcal{U} \psi$, contrary to our assumption. Thus, we have (2).

Let us show now that

$$\vdash \vartheta \to \bigcirc\vartheta. \tag{3}$$

If this is not the case then the formula $\vartheta \wedge \bigcirc\neg\vartheta$ is consistent, and so there is $\mathfrak{D} \in \mathscr{S}$ such that $\beta_{\mathfrak{D}} \wedge \bigcirc\neg\vartheta$ is consistent. By Lemma 10(i) and (ii), we have a pointed state candidate $\mathfrak{C}$ for which $\mathfrak{D} \prec \mathfrak{C}$. But then $\mathfrak{C} \in \mathscr{S}$ and $\beta_{\mathfrak{D}} \wedge \bigcirc\beta_{\mathfrak{C}}$ is consistent, contrary to consistency of $\beta_{\mathfrak{D}} \wedge \bigcirc \bigwedge_{\mathfrak{C}\in\mathscr{S}} \neg\beta_{\mathfrak{C}}$. Thus, we have (3).

It follows from (2), (3), and (u_2) that $\vdash \vartheta \to \neg(\varphi\mathscr{U}\psi)$. As $\beta_{\mathfrak{P}_0}$ is a disjunct of ϑ, we then have

$$\vdash \beta_{\mathfrak{P}_0} \to \neg(\chi\mathscr{U}\psi)$$

contrary to (1).

The existence of a c-realizing sequence is proved analogously. □

Lemma 12. *Suppose φ is a monodic sentence consistent with $\mathscr{MON}$. Then φ is satisfied in a quasi-model for φ.*

Proof. Let π_φ be the disjunction of formulas $\beta_{\mathfrak{P}}$, for all pointed state candidates for φ. Clearly, $\vdash \pi_\varphi$, and so $\varphi \wedge \pi_\varphi$ is consistent. Then we have a consistent pointed state candidate $\langle\mathfrak{C}_0, t_0\rangle$ such that $\varphi \in t_0$. $\mathfrak{C}_0$ will be the starting state candidate in the quasi-model $\mathfrak{Q} = (\mathfrak{C}_i = \langle T_i, T_i^{con}\rangle\colon i \in \mathbb{N})$ to be constructed.

Take some $t \in T_0$ and $\chi\mathscr{U}\psi \in t$. The pointed state candidate $\langle\mathfrak{C}_0, t\rangle$ is clearly consistent. So, by Lemma 11, there is a sequence of pointed state candidates

$$\langle\mathfrak{C}_0, t\rangle \prec \langle\mathfrak{C}_1, t_1\rangle \prec \cdots \prec \langle\mathfrak{C}_k, t_k\rangle \tag{4}$$

realizing $\chi\mathscr{U}\psi$ in t. Next we take another formula $\chi'\mathscr{U}\psi' \in t$, if any, which is not realized in this sequence. In this case, by Lemma 9(i), we have $\chi'\mathscr{U}\psi' \in t_k$. Using Lemma 11 once again, we extend (4) to

$$\langle\mathfrak{C}_0, t\rangle \prec \langle\mathfrak{C}_1, t_1\rangle \prec \cdots \prec \langle\mathfrak{C}_k, t_k\rangle \prec \cdots \prec \langle\mathfrak{C}_l, t_l\rangle \tag{5}$$

realizing $\chi'\mathscr{U}\psi'$ in t. Following this way, we can construct a sequence of the form (5) realizing all formulas of the form $\chi\mathscr{U}\psi$ in t. Let (5) be such a sequence.

Now take another type $t' \in T_0$. By Lemma 10(ii), there are types $t_i' \in T_i$, $0 < i \leqslant l$, such that

$$\langle\mathfrak{C}_0, t'\rangle \prec \langle\mathfrak{C}_1, t_1'\rangle \prec \cdots \prec \langle\mathfrak{C}_l, t_l'\rangle.$$

In precisely the same manner as before we extend this sequence to realize all formulas of the form $\chi\mathscr{U}\psi$ in t'. After that we consider yet another type $t'' \in T_0$, and so forth. When all types are exhausted, we will have a sequence of state candidates $\mathfrak{C}_0, \ldots, \mathfrak{C}_n$. (If no type in $\mathfrak{C}_0$ contains formulas of the form $\chi\mathscr{U}\psi$, we take a state candidate $\mathfrak{C}_1$ such that the pair $(\mathfrak{C}_0, \mathfrak{C}_1)$ is suitable and put $\mathfrak{C}_n = \mathfrak{C}_1$.)

We did not yet take care of the constants. So suppose $\langle t, c\rangle \in T_0^{con}$ and $\chi\mathscr{U}\psi \in t$. Pick the $t_i \in T_i$ with $\langle t_i, c\rangle \in T_i^{con}$, for $1 \leqslant i \leqslant n$. By Lemma 10(iii) we have

$$\langle\mathfrak{C}_0, t\rangle \prec_c \langle\mathfrak{C}_1, t_1\rangle \prec_c \cdots \prec_c \langle\mathfrak{C}_n, t_n\rangle. \tag{6}$$

If $\varphi\mathcal{U}\psi$ is not c-realized by this sequence in t, then $\varphi\mathcal{U}\psi \in t_n$. By Lemma 10(i) and (iii) we can extend (6) to

$$\langle \mathfrak{C}_0, t\rangle \prec_c \langle \mathfrak{C}_1, t_1\rangle \prec_c \cdots \prec_c \langle \mathfrak{C}_n, t_n\rangle \prec_c \cdots \prec_c \langle \mathfrak{C}_{n'}, t_{n'}\rangle \tag{7}$$

with $\psi \in t_{n'}$. Next we take another $\langle t', d\rangle \in T_0^{con}$ and $\chi'\mathcal{U}\psi' \in t'$ which is not d-realized by the sequence

$$\langle \mathfrak{C}_0, t'\rangle \prec_d \langle \mathfrak{C}_1, t_1'\rangle \prec_d \cdots \prec_d \langle \mathfrak{C}_n, t_n'\rangle \prec_d \cdots \prec_d \langle \mathfrak{C}_{n'}, t_{n'}'\rangle, \tag{8}$$

where t_i' is the point with $\langle t_i', d\rangle \in T_i^{con}$, $1 \leqslant i \leqslant n'$. We extend (8) so that $\chi'\mathcal{U}\psi'$ is d-realized in the new sequence. After that we consider yet another pair $\langle t'', c''\rangle$ and $\chi''\mathcal{U}\psi''$, and so forth. When all pairs $\langle t, c\rangle \in T_0^{con}$ and all $\chi\mathcal{U}\psi \in t$ are exhausted, we have a sequence $\mathfrak{C}_0, \ldots, \mathfrak{C}_m$.

Then we consider the types and constants from $\mathfrak{C}_m$ and construct a sequence $\mathfrak{C}_m, \ldots, \mathfrak{C}_{m'}$ as if $\mathfrak{C}_m$ were $\mathfrak{C}_0$. After that we take care of $\mathfrak{C}_{m'}$, and so on.

It is readily seen (using Lemma 9) that the resulting infinite sequence $\mathfrak{Q}$ is a quasi-model for φ. $\Box$

This completes the proof of Theorem 2.

3. Monodicity and equality

So far we have considered the first-order temporal language $\mathcal{TL}$ *without equality and functional symbols*. A natural question is whether our decidability and axiomatizability results concerning the class of monodic formulas can be generalized to the language with these ingredients. It should be clear that functional symbols easily destroy nice properties of the monodic formulas: in the proof of Theorem 2 from [8] we can replace $Q_2(y)$ and $P_j(y)$ with $Q_2(f(x))$ and $P_j(f(x))$, respectively, thus obtaining a monodic–monadic one-variable formula $\varphi_{\mathfrak{T}}$, associated with a finite set of tiles $\mathfrak{T}$, such that $\varphi_{\mathcal{T}}$ is satisfiable iff $\mathfrak{T}$ recurrently tiles $\mathbb{N} \times \mathbb{N}$. So the class of such formulas cannot be recursively enumerable.

In this section we show that by adding equality we also 'spoil' the monodic fragment. Namely, we are going to prove that the monodic fragment $\mathcal{TL}_1^{=}$ *with equality* is not recursively axiomatizable.

Let us fix a unary predicate P and denote by χ the conjunction of the following formulas:

$$\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y), \tag{9}$$

$$\Box_{\mathrm{F}}^{+} \forall x (P(x) \rightarrow \bigcirc P(x)), \tag{10}$$

$$\Box_{\mathrm{F}}^{+} \forall x \forall y (\bigcirc P(x) \wedge \bigcirc P(y) \wedge \neg P(x) \wedge \neg P(y) \rightarrow x = y), \tag{11}$$

$$\Diamond_{\mathrm{F}} \forall x (P(x) \leftrightarrow \Diamond_{\mathrm{F}} P(x)). \tag{12}$$

Here $\Box_{\mathrm{F}}^{+}$ means 'now and always in the future', i.e., $\Box_{\mathrm{F}}^{+}\varphi = \neg(\top\mathcal{U}\neg\varphi)$.

The reader can readily check that the following lemma holds.

Lemma 13. *For every model* $\mathfrak{M} = \langle\langle \mathbb{N}, <\rangle, D, I\rangle$, *we have* $0 \models \chi$ *iff the following conditions are satisfied*:

- $|P^0| = 1$;
- $\forall n \in \mathbb{N}\ (P^n \subseteq P^{n+1}\ \&\ |P^{n+1} - P^n| \leqslant 1)$;
- $\exists m \in \mathbb{N}\ \forall k \geqslant m\ P^m = P^k$.

(In other words: There is a unique element $a_0 \in D$ for which $P(a_0)$ holds true at moment 0; $P(a_0)$ remains true always in the future. At moment 1 there may be only two elements $a_0, a_1 \in D$ for which P is true, at moment 2 only three such elements, etc. Finally, we eventually reach a moment m starting from which P is stable.)

Suppose now that we are given an arbitrary first-order (non-temporal) sentence ψ which does not contain occurrences of P. Let Q be a unary predicate not occurring in ψ either. Put

$$\chi' = \forall x (Q(x) \leftrightarrow \Diamond P(x))$$

and denote by ψ^Q the relativization of ψ to Q (i.e., $\varphi^Q = \varphi$ for atomic φ, ^{Q} commutes with the Booleans, and $(\forall x\ \varphi)^Q = \forall x\ (Q(x) \to \varphi^Q)$). Clearly, all the formulas χ, χ', and ψ^Q are in $\mathscr{TL}_1^=$.

Lemma 14. *The following conditions are equivalent*:
- ψ *is valid in all finite classical first-order models*;
- $\chi \wedge \chi' \to \psi^Q$ *is valid in all temporal first-order models based on* $\langle \mathbb{N}, <\rangle$.

Proof. Suppose $\chi \wedge \chi' \to \psi^Q$ is refuted in $\langle\langle \mathbb{N}, <\rangle, D, I\rangle$. Without loss of generality we may assume that $0 \models \chi \wedge \chi'$ and $0 \not\models \psi^Q$. By Lemma 13, Q^0 (the set of elements in D on which Q is true at moment 0) is finite. Let M be the classical first-order model with domain Q^0 and the $n$ary predicates R^M, $n \geqslant 0$, defined by taking, for every n-tuple $a_1, \ldots, a_n$ of elements in Q^0,

$$(a_1, \ldots, a_n) \in R^M \quad \text{iff} \quad (a_1, \ldots, a_n) \in R^0.$$

It is easily checked by induction that for every assignment $\mathfrak{a}$ in Q^0 and every classical first-order formula ϑ, we have $0 \models^{\mathfrak{a}} \vartheta^Q$ iff $M \models^{\mathfrak{a}} \vartheta$. It follows that the finite model M refutes ψ.

Conversely, suppose that M is a finite first-order model refuting ψ and having domain $D = \{a_0, \ldots, a_n\}$. Define a temporal model $\langle\langle \mathbb{N}, <\rangle, D, I\rangle$ in such a way that $I(0)$ interprets the predicate symbols in ψ by the same predicates as in M, $Q^0 = D$, and for every $i \in \mathbb{N}$,

$$P^i = \begin{cases} \{a_0, \ldots, a_i\} & \text{if } i \leqslant n, \\ D & \text{if } i > n. \end{cases}$$

It follows from Lemma 13 that $0 \models \chi \wedge \chi'$, and clearly we have $0 \not\models \psi^Q$. $\square$

Now recall that by Trakhtenbrot's theorem (see, e.g. [1]) the set of first-order classical formulas that are valid in finite models is not recursively enumerable. As a consequence we obtain the following:

Theorem 15. *The set of $\mathcal{TL}_1^=$-formulas that are valid in all temporal models based on $\langle \mathbb{N}, < \rangle$ is not recursively enumerable, and so not recursively axiomatizable.*

It is not clear, however, whether the decidable fragments of first-order temporal logics found in [8] and below remain decidable after extending the language with equality. [1]

4. Two more decidable fragments

As was shown in [8], the two-variable monodic-fragment, the monadic–monodic fragment, and the guarded monodic fragment of many first-order temporal logics are decidable. Here we extend this list by observing that the monodic fragment can be naturally combined with the fluted fragment of classical first-order logic, which was shown to be decidable and to have the finite model property in [9,10].

Let $X_m = (x_1, \ldots, x_m)$ be a list of individual variables.

Definition 16 (*Fluted formulas*). An *atomic fluted formula over X_i* is an atom of the form $P(x_k, x_{k+1}, \ldots, x_i)$ for some $k \leqslant i$. *Fluted formulas* are now defined inductively as follows:

- any atomic fluted formula over X_i is a fluted formula over X_i;
- if φ is a fluted formula over X_{i+1}, then both $\exists x_{i+1}\varphi$ and $\forall x_{i+1}\varphi$ are fluted formulas over X_i;
- any Boolean combination of fluted formulas over X_i is a fluted formula over X_i;
- if φ and ψ are fluted formulas over X_i then $\varphi \mathcal{U} \psi$, $\varphi \mathcal{S} \psi$, $\bigcirc\varphi$, and $\bigcirc_{\mathrm{P}}\varphi$ are fluted formulas over X_i.

Finally, we say a formula is *fluted* if it is fluted over X_i for some $i \in \mathbb{N}$.

Denote by $\mathcal{FLU}$ the set of all fluted formulas of our first-order temporal language.

Theorem 17. *Let $\mathcal{F}$ be any of the following classes of flows of time*:

- $\{\langle \mathbb{N}, < \rangle\}$;
- $\{\langle \mathbb{Z}, < \rangle\}$;
- $\{\langle \mathbb{Q}, < \rangle\}$;
- *the class of all finite strict linear orders*;
- *any first-order-definable class of strict linear orders*,

[1] We have been just informed by Degtyarev and Lisitsa that the (decidable) monadic two-variable fragment of $\mathcal{TL}_1$ becomes not recursively enumerable if extended with equality.

and let $\mathscr{F}^+$ *range over these and* $\{\langle \mathbb{R}, < \rangle\}$. *Then the fragments*

$$TL(\mathscr{F}) \cap \mathscr{T}\mathscr{L}_1 \cap \mathscr{F}\mathscr{L}\mathscr{U} \quad \textit{and} \quad TL_{fin}(\mathscr{F}^+) \cap \mathscr{T}\mathscr{L}_1 \cap \mathscr{F}\mathscr{L}\mathscr{U}$$

are decidable.

Proof. In view of Theorems 15, 36 and Corollary 37 of [8], and the results of [9,10], it is enough to show that if φ is a fluted monodic formula and $\mathfrak{C} = \langle T, T^{con} \rangle$ a state candidate for φ, then the formula

$$\bigwedge_{t \in T} \exists x\, \bar{t}(x) \quad \wedge \quad \forall x \bigvee_{t \in T} \bar{t}(x) \quad \wedge \bigwedge_{\langle t,c \rangle \in T^{con}} \bar{t}(c)$$

is equivalent to a fluted sentence. But this is almost obvious: by only renaming variables in formulas from $\overline{sub_x\, \varphi}$, we can rewrite all of them as fluted (classical) formulas over X_1 with at most one free variable x_1. $\square$

As was observed by Hodkinson (see [7]), Theorem 74 of [8] can be extended to the *loosely guarded fragment* introduced in [12].

Definition 18 (*Loosely guarded fragment*). Denote by $\mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$ the smallest set of $\mathscr{T}\mathscr{L}$-formulas such that

- every atomic formula is in $\mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$;
- if φ and ψ are in $\mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$, then so are $\varphi \wedge \psi$, $\neg\varphi$, $\varphi \mathscr{S} \psi$, $\varphi \mathscr{U} \psi$, $\bigcirc\varphi$, and $\bigcirc_{\mathrm{P}}\varphi$;
- if
  - γ is a conjunction of atoms,
  - $\varphi \in \mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$,
  - every free variable of φ occurs in γ, and
  - for some tuple $\bar{y}$ of variables in γ, if $y \in \bar{y}$ and x is a variable in γ different from y then there is a conjunct of γ containing both x and y, then $\exists \bar{y}(\gamma \wedge \varphi) \in \mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$.

The set $\mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$ is called the *loosely guarded fragment* of the first-order temporal language.

Using the fact that the loosely guarded fragment of classical first-order logic is decidable [12,6] and has the finite model property [7], and following the proof of Theorem 74 of [8], one can readily prove the following:

Theorem 19. *Let* $\mathscr{F}$ *and* $\mathscr{F}^+$ *be as in Theorem* 17. *Then the fragments*

$$TL(\mathscr{F}) \cap \mathscr{T}\mathscr{L}_1 \cap \mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F} \quad \textit{and} \quad TL_{fin}(\mathscr{F}^+) \cap \mathscr{T}\mathscr{L}_1 \cap \mathscr{T}\mathscr{L}\mathscr{G}\mathscr{F}$$

are decidable.

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References

[1] E. Börger, E. Grädel, Yu. Gurevich, The Classical Decision Problem, Perspectives in Mathematical Logic, Springer, Berlin, 1997.
[2] J. Chomicki, D. Niwinski, On the feasibility of checking temporal integrity constraints, J. Comput. Systems Sci. 51 (1995) 523–535.
[3] J. Chomicki, D. Toman, Temporal logic in information systems, in: J. Chomicki, G. Saake (Eds.), Logics for Databases and Information Systems, Kluwer Academic Publishers, Dordrecht, 1998, pp. 31–70.
[4] D. Gabbay, I. Hodkinson, M. Reynolds, Temporal Logic: Mathematical Foundations and Computational Aspects, vol. 1, Oxford University Press, Oxford, 1994.
[5] R.I. Goldblatt, Logics of Time and Computation, CSLI Lecture Notes, vol. 7, Stanford, CSLI, 1987.
[6] E. Grädel, Decision procedures for guarded logics, Automated Deduction — CADE-16, Proceedings of the 16th International Conference on Automated Deduction, Lecture Notes in Artificial Intelligence, vol. 1632, Springer, Berlin, 1999, pp. 31–51.
[7] I. Hodkinson, Loosely guarded fragment of first-order logic has the finite model property, Stud. Logica, 2001, in print.
[8] I. Hodkinson, F. Wolter, M. Zakharyaschev, Decidable fragments of first-order temporal logics, Ann. Pure Appl. Logic 106 (2000) 85–134.
[9] Z. Manna, A. Pnueli, The Temporal Logic of Reactive and Concurrent Systems, Springer, Berlin, 1992.
[10] W.C. Purdy, Decidability of fluted logic with identity, Notre Dame J. Formal Logic 37 (1996) 84–104.
[11] W.C. Purdy, Fluted formulas and the limits of decidability, J. Symbolic Logic 61 (1996) 608–620.
[12] J. van Benthem, Dynamic bits and pieces. Technical Report LP-97-01, ILLC, University of Amsterdam, 1997.